Fixed costs and the axiomatization of Shapley's sharing rule

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Abstract

The cost sharing rule derived from the Shapley value is the unique sharing rule which allocates fixed costs uniformly.

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1 Introduction

There is a large literature on cost sharing based on solutions defined for cooperative games with transferable utility.[1] The allocation of a fixed cost is however not explicitly addressed. Fairness suggests that a fixed cost should be uniformly allocated among the players it affects. That is actually what does the most popular sharing rule which is derived from the Shapley value.

The nucleolus[2] induces another sharing rule proposed in the literature and used in actual cost sharing problems.[3] Examples show however that it does not necessarily allocate a fixed cost uniformly. Actually, the Shapley sharing rule is the only sharing rule which does so. This offers an alternative axiomatization which does not include additivity.

2 Cost games

Given a set $N = \{1...n\}$ of players, a *cost sharing game* is defined by a cost function C which associates to each coalition $S \subset N$ a cost $C(S) \in I\!R$. By convention, $C(\emptyset) = 0$. The set of cost functions defined on the set of players N is denoted by $G(N)$. Any linear combination of cost functions on a common set of players N is a cost function. As a consequence, $G(N)$ is a linear space of dimension $2^n - 1$.

Notation. The letters n, s, $t, ...$ will denote the size of the sets N, S, $T, ...$ For any vector x, $x(S)$ will denote the sum over S of its coordinates. Coalitions will be identified as $ijk...$ instead of $\{i,\ j,\ k, ...\}$.

A cost function C is *subadditive* if $S \cap T = \emptyset$ implies $C(S) + C(T) \geq C(S \cup T)$. A cost function C is *symmetric* if the cost associated to a coalition depends only on its size: $C(S) = g(s)$ for some function g satisfying $g(0) = 0$.

A *cost sharing rule* is a mapping R which associates to any cost function $C \in G(N)$ an allocation $y = R(C) \in I\!R^n$ such that $y(N) = C(N)$.

[1] See for instance Moulin (1988) or Young (1985).

[2] See Schmeidler (1969).

[3] The relations between the Shapley value, the core and the nucleolus in concave cost games is studied in Dehez and Vanden Eynde (2006).

3 Fixed costs

Basically a fixed cost $f \in I\!R$ is a cost which affects all players and coalitions equally:

$$C_f(S) = C(S) + f \quad \text{for all } S \subset N, S \neq \emptyset \tag{1}$$

with $C_f(\emptyset) = 0$.[4] Adding a fixed cost is not a simple translation: f is a cost which enters in the computation of the costs associated to all *nonempty* coalitions. If only a subset $T \subset N$ of players are concerned, f qualifies as a fixed cost if it also affects all coalitions including players in T:

$$\begin{aligned} C_f(S) &= C(S) + f \quad && \text{for all } S \subset N \text{ such that } S \cap T \neq \emptyset \\ &= C(S) && \text{for all } S \subset N \backslash T \end{aligned} \tag{2}$$

Clearly (1) is a particular case of (2) for $T = N$. In all cases the amount to be allocated is equal to $C(N) + f$.

How should a fixed cost be allocated ? Fairness requires that only the players who are concerned contribute and that they contribute equally:

Fairness For any cost function $C \in G(N)$, fixed cost $f \in I\!R$ and subset $T \subset N$,

$$\begin{aligned} R_i(C_f) &= R_i(C) + \tfrac{1}{t} f \quad && \text{for all } i \in T \\ &= R_i(C) && \text{for all } i \in N \backslash T \end{aligned}$$

where C_f is given by (2).

4 The Shapley value

Applied to a cost function $C \in G(N)$, the Shapley value induces a sharing rule φ which allocates to each player i a weighted sum of his or her marginal costs:

$$\varphi_i(C) = \sum_{S \subset N} \alpha(s)[C(S) - C(S \backslash i)] \tag{3}$$

where the weights $\alpha(s)$ are given by

$$\alpha(s) = \frac{(s-1)!(n-s)!}{n!}.$$

[4] Costs as well as fixed cost can be negative. Fixed costs can alternatively be interpreted as taxes when positive and as subsidies when negative.

The following axioms define uniquely Shapley's sharing rule[5]:

Efficiency	For all $C \in G(N)$, $\sum_i R_i(C) = C(N)$.
Anonymity	For all $C \in G(N)$ and all permutations π of N, $R_j(\pi C) = R_i(C)$ for all $i \in N$, where $j = \pi(i)$ and πC is defined by $\pi C(\pi S) = C(S)$.
Dummy	For all $C \in G(N)$, $C(S) = C(S \backslash i)$ for all $S \subset N$ implies $R_i(C) = 0$.
Additivity	For all C_1, C_2 in $G(N)$, $R(C_1 + C_2) = R(C_1) + R(C_2)$.

Shapley shares are individually rational for subadditive cost functions: $\varphi_i(C) \leq C(i)$ for all i. However there may be cross-subsidization because either the core is empty or because the allocation does not belong to the core.[6]

Lemma *The Shapley sharing rule satisfies the fairness axiom.*

Proof. This is a direct consequence of Shapley' axioms. Let T be the set of players affected by a fixed cost f and consider the game C_{of} defined by (2) with $C(S) = 0$ for all $S \subset N$. The cost function C_{of} defines a game where players in T are symmetric and players outside T are dummies. The anonymity and dummy axioms imply that

$$\begin{aligned} \varphi_i(N,\ C_{of}) \quad &= \tfrac{1}{t} f \quad \text{for all } i \in T \\ &= 0 \quad \text{for all } i \in N \backslash T \end{aligned}$$

Fairness then follows from additivity. ∎

The following examples show that the sharing rule derived from the nucleolus does not necessarily allocate a fixed cost uniformly.

Example 1. Consider the 3-player cost game defined by $C(1) = 6$, $C(2) = 7$, $C(3) = 10$, $C(12) = 9$, $C(13) = 13$, $C(23) = 15$ and $C(123) = 16$. The cost allocations derived from the Shapley value and the nucleolus are given by (3.17, 4.67, 8.17) and (2.75, 4.75, 8.50) respectively. Consider a fixed cost $f = 3$ affecting all players and coalitions. As expected, the Shapley value imposes to each player an additional contribution equal to 1 while the nucleolus imposes additional contributions given by (0.92, 0.92, 1.17).

Example 2. Consider the cost game defined by $C(1) = 7$, $C(2) = 9$, $C(3) = 13$, $C(4) = 15$, $C(12) = 15$, $C(13) = 19$, $C(14) = 21$, $C(23) = 20$, $C(24) = 22$, $C(34) = 25$, $C(123) = 27$, $C(124) = 26$, $C(134) = 24$, $C(234) = 26$

[5] See Shapley (1953, 1971).

[6] The core is the set of allocations y such that $y(N) = C(N)$ and $y(S) \leq C(S)$ or, equivalently, $y(S) \geq C(N) - C(N \backslash S)$ for all $S \subset N$. See Faulhaber (1975).

and $C(1234) = 30$. The cost allocations derived from the Shapley value and the nucleolus are given by (5.08, 6.75, 8.58, 9.58) and (5.50, 7.50, 8.25, 8.75) respectively. For a fixed cost $f = 2$ affecting players 1 and 2, ie $T = \{1, 2\}$, the contributions of player 1 and 2 increase by 1 in the Shapley allocation — as expected — while the nucleolus imposes additional contributions to all four players given by (0.83, 0.83, 0.25, 0.08).

Proposition *The Shapley sharing rule is the unique sharing rule satisfying the dummy and fairness axioms.*

Proof. Consider a sharing rule R satisfying the dummy and fairness axioms. For any given $T \subset N$, $T \neq \emptyset$, define the elementary game e_T by[7]

$$\begin{array}{ll} e_T(S) = 1 & \text{if} \quad S \cap T \neq \emptyset \\ e_T(S) = 0 & \text{if} \quad S \subset N \backslash T \end{array}$$

These $p = 2^n - 1$ games are linearly independent:

$$\sum_{T \cap S \neq \emptyset} \alpha_T = 0 \;\text{ for all } S \subset N \;\text{ implies } \alpha_T = 0 \;\text{ for all } T \subset N, T \neq \emptyset$$

Hence they form a basis[8] of the p-dimensional linear space $G(N)$: for every cost function $C \in G(N)$ there exists a unique vector $\alpha \in \mathbf{R}^p$ such that

$$C(S) = \sum_{T \neq \emptyset} \alpha_T e_T(S) \equiv \sum_{T \neq \emptyset} e'_T(S)$$

where the games e'_T are defined by

$$\begin{array}{lll} e'_T(S) & = \alpha_T & \text{if } S \cap T \neq \emptyset \\ & = 0 & \text{if } S \subset N \backslash T \end{array}$$

Players outside T are dummies. Applying the dummy and fairness axioms we get:

$$\begin{array}{rcll} R_i(e'_T) & = & \frac{1}{t}\alpha_T & \text{for all } i \in T \\ & = & 0 & \text{for all } i \in N \backslash T \end{array}$$

The fairness axiom then ensures that R can be extended to the all space $G(N)$ in a unique way:

$$R_i(C) = \Sigma_{T \neq \emptyset} R_i(e'_T).$$

It must therefore coincide with the Shapley sharing rule. ■

[7] These are simple games describing a decision problem where a coalition is winning if and only if it contains *at least one* member of T.

[8] The basis used by Shapley is different. He uses simple games where the players in T are veto players: a coalition is winning if and only if it contains *all* members of T. This gives rise to a $p \times p$ matrix whose determinant is equal to 1, instead of -1 in our formulation.

Actually the dummy axiom can be replaced by an axiom fixing the origin:[9]

Zero cost $C(S) = 0$ for all $S \subset N$ implies $R(C) = 0$.

Efficiency is not formally mentioned because it is part of the definition of a sharing rule. Anonymity is not needed explicitly: it is included in the fairness axiom. The only limitation of our axiomatic is that the result does not hold when restricted to the cone of subadditive cost games.

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